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Partition Function for an Electron in a Random Potential

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We compute the average partition function for an electron moving in a Gaussian random potential. A path integral formulation is used, with a trial action like that in Feynman's polaron theory. We compute the variational bound as well as the first correction in a systematic cumulant expansion. The results are checked against exact formulas for the one-dimensional white noise problem. The density of states in the low-energy tail has the correct exponential energy dependence, and energy-dependent prefactor to within a few percent. In addition, the partition function goes over smoothly to the perturbation theory result at high temperatures.

KEY WORDS: Random potential; density of states; partition function.

1. INTRODUCTION

We consider an electron moving in a potential $V(r)$, which is a random field obeying Gaussian statistics. We are interested in computing the partition function averaged over the random field configurations. The density of states may be found as the inverse Laplace transform of the partition function.

This problem was studied by Frisch and Lloyd[3] and Halperin,[2] who exhibited the exact solution for the density of states for white noise in one dimension. The general physical features of the three-dimensional case, with particular attention to the low-energy tail in the density of states, were elucidated by Halperin and Lax[3] and Zittarz and Langer.[4] For general reviews consult Refs. 5–7.

In the present paper we use the path integral representation of the partition function and make a calculation very similar to Feynman's theory of the polaron.[8] For that problem, Feynman obtained an excellent approximation to the ground-state energy and effective mass over the entire range of coupling constants. He used a two-time quadratic trial action. For the

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present problem we compute the partition function as a smoothly varying quantity over the entire temperature range. The electron in a random potential is a simpler system than the polaron. As a result we obtain very explicit results in the Feynman approximation. In addition, we study the structure of a systematic theory based on a cumulant expansion which starts with the Feynman approximation as the first step. A theory of this type for a more difficult random impurity problem was explored by Friedberg and Luttinger.[9] They used a single time trial action and studied the low-energy tail in the density of states.

To see the main features of the problem at low temperatures most directly, one can employ a delightful variational principle invented by Luttinger.[10] This principle does not aim for great accuracy, but permits a quick overview of a large number of problems. Luttinger shows that for an electron with a Hamiltonian

$$H = \tfrac{1}{2}p^2 + V(r) \tag{1}$$

the canonical partition function has the lower bound

$$Z = \operatorname{tr}(e^{-\beta H}) > (2\pi\beta)^{-\epsilon/2} \int d_\epsilon Q \exp[-\beta\langle\psi/V(r + Q)/\psi\rangle - \tfrac{1}{2}\beta\langle\psi, p^2\psi\rangle + \tfrac{1}{2}\beta\langle\psi, p\psi\rangle^2] \tag{2}$$

Here $\psi(r)$ is an arbitrary, normalized, spatially localized state, which serves to generate a coherent-state-type representation.[2] The index ϵ refers to the space dimensionality.

The random field $V(r)$ may be expanded in an arbitrary orthonormal basis $\psi_m(r)$

$$V(r) = \sum_{m=0}^{\infty} \xi_m \psi_m(r) \tag{3}$$

The assumption of a Gaussian random potential means that the average of a physical quantity such as Z is given by the functional average:

$$\langle Z\rangle = \int \delta V(r) Z \exp\left[-\tfrac{1}{2}\iint V(r)W^{-1}(r - r^1)V(r^1)\, d_\epsilon r\, d_\epsilon r^1\right] \tag{4}$$

with

$$\langle V(r)V(r^1)\rangle = W(r - r^1) \tag{5}$$

In particular, for Gaussian white noise in one dimension we have

$$\langle Z\rangle = \int Z\left[\exp -\left(\sum \frac{\xi_m^2}{2\gamma}\right)\right] \prod_m \frac{d\xi_m}{\gamma\sqrt{2\pi}} \tag{6}$$

$$\langle V(r)V(r^1)\rangle = \gamma\, \delta(r - r^1) \tag{7}$$

[2] This type of lower bound, based on coherent states, was used for earlier spin systems.[11]

where γ measures the strength of the noise. We use the completeness of the functions $\psi_m(r)$ to transform the Luttinger principle to

$$\langle Z\rangle/L > (2\pi\beta)^{-1/2}\exp\left(-\tfrac{1}{2}\beta\langle\psi, p^2\psi\rangle + \tfrac{1}{2}\beta^2\gamma\int\psi^4\,dr\right) \tag{8}$$

where L is the length of the container. The function $\psi(r)$ is at our disposal (for each value of β) to maximize the value of the exponent. For one-dimensional white noise the exact result for $\psi(r)$ is

$$\psi(r) = (2a)^{-1/2}\operatorname{sech}(r/a), \qquad a = 2/\beta\gamma \tag{9}$$

This yields

$$\langle Z\rangle/L > (2\pi\beta)^{-1/2}\exp(\gamma^2\beta^3/24) \tag{10}$$

One can do this calculation in two steps. First introduce a family of normalized functions $\psi(r) = a^{-1/2}\psi_0(r/a)$. Then the variational principle leads to

$$a = (2/\beta\gamma)\int\psi_0 p^2\psi_0\,dr\Big/\int\psi_0{}^4\,dr \tag{11}$$

The choice of an oscillator function $\psi_0 = \pi^{-1/4}\exp(-r^2/2)$ leads to a slightly inferior bound where 24 is replaced by 8π, i.e., an error of 5% in the coefficient of the exponential.

The Luttinger bound holds for all β. The density of states in the low-energy tail is obtained by a saddle point approximation to the inverse Laplace transform. The saddle point is located at $\beta_0 = (8|E|)^{1/2}/\gamma$. It leads to a limiting density of states proportional to

$$|E|^{-1/2}\exp\left(-\frac{4\sqrt{2}}{3}|E|^{3/2}\right)$$

This can be compared with the exact result

$$\frac{4}{\pi}\frac{1}{\gamma}|E|\exp\left(-\frac{4\sqrt{2}}{3}|E|^{3/2}\right) \tag{12}$$

The coefficient of the dominant exponential term is exact, but the prefactor has a different energy dependence. We will discuss the finer details of the low-energy tail later.

In the high-temperature region the Luttinger bound is poor. It describes the effect of the mean potential on the electron but not the perturbation-theoretic corrections. The exponential actually has a $\beta^{3/2}$ dependence rather than β^3. This is most easily obtained in the path integral formulation, to which we now turn.

2. PATH INTEGRAL FORM OF PARTITION FUNCTION

The partition function for a particle of unit mass moving in a potential $V(r)$ is given by the path integral[12]

$$Z = \int d_\epsilon r_0 \int_{r_0} e^{-\mathscr{S}} \mathscr{D}r(u) \tag{13}$$

where the paths $r(u)$ start and end at a point r_0. The final integration $d_\epsilon r_0$ goes over the region occupied by the system. The action is

$$\mathscr{S} = \tfrac{1}{2} \int_0^\beta \dot{r}^2(u)\, du + \int_0^\beta V(r(u))\, du \tag{14}$$

For a Gaussian random process the averaged partition function is

$$\langle Z \rangle = \int d_\epsilon r_0 \int_{r_0} e^{-S} \mathscr{D}r(u) \tag{15}$$

with

$$S = \tfrac{1}{2} \int \dot{r}^2(u)\, du - \tfrac{1}{2} \int_0^\beta \int_0^\beta W(r(u) - r(u'))\, du\, du'$$

The ease with which an exact result is obtained for the averaged partition function is one of the most attractive and powerful features of path integral methods. The difficulties are of course transferred to the evaluation of a functional integral involving a two-time action. This functional is simpler than the one that enters in the polaron problem, where W also depends explicitly on the difference $u - u'$. This arises physically from the inertial lag of the phonon response. The functional is also simpler than the one found in the random impurity problem treated by Edwards and Gulyaev,[13] Jones and Lukes,[14] Friedberg and Luttinger,[9] Freed,[15] and others. The above functional arises as the limit of high density and weak scatterers.

Nevertheless, as emphasized by Freed, if there are universal features of the random impurity problem, they will appear in the study of Eq. (15). In addition, there are exact results known for one dimension[1,2] and approximate treatments[3,4] of the functional integral that yield exact results for the low-energy tails in the density of states. Thus, a thorough treatment of the Gaussian noise problem is a good testing ground for general approximation schemes.

A large number of papers treat the two-time action by using Feynman's variation principle. For each point r_0 we have the upper bound

$$\int_{r_0} e^{-S} \mathscr{D}r(u) > \int_{r_0} e^{-S} \mathscr{D}r(u) \exp(\langle S - S \rangle_{S_0, r_0}) \tag{16}$$

with

$$\langle S - S_0 \rangle_{S_0, r_0} = \int_{r_0} (S - S_0)\, e^{-S_0} \mathscr{D}r(u) \Big/ \int_{r_0} e^{-S_0} \mathscr{D}r(u')$$

S is some real time trial action, with parameters or functional forms chosen to optimize the bound. Jones and Lukes and Friedberg and Luttinger note that one can take this as the starting point of a complete theory in terms of cumulants. The exponent becomes

$$\langle S - S_0\rangle + (1/2!)[\langle(S - S_0)^2\rangle - \langle S - S_0\rangle^2] + \cdots.$$

With this approach attention shifts to a choice of S_0, which should embody as much of the physics of S as possible, but for which the functional averages can be performed.

The simplest choice for S_0 is the free particle action for which

$$\int_{r_0} e^{-S_0}\,\mathscr{D}r(u) = (2\pi\beta)^{-\epsilon/2} \tag{17}$$

We also have the formula

$$\left\langle \exp\int_0^\beta f(t)r(t)\,dt \right\rangle_{S_0 r_0} = \exp(+P)$$

$$P = +\frac{r_0}{2}\int_0^\beta f(u)\,du - \frac{1}{2}\int_0^\beta f(u)\,du\int_0^u dt_1\int_0^{t_1} f(t_2)\,dt_2 \tag{18}$$

$$+\frac{1}{2}\int_0^\beta \frac{f(u)u\,du}{\beta}\int_0^\beta dt_1\int_0^{t_1} f(t_2)\,dt_2$$

Let

$$A = \tfrac{1}{2}\int_0^\beta\int_0^\beta W(r(u) - r(u'))\,du\,du'$$

$$= \tfrac{1}{2}(2\pi)^{-\epsilon}\int d_\epsilon k\; W(k)\int_0^\beta\int_0^\beta e^{ik[r(t)-r(s)]}\,dt\,ds \tag{19}$$

With $f(u) = ik[\delta(u - t) - \delta(u - s)]$, we find

$$\langle A\rangle = \tfrac{1}{2}\beta^2(2\pi)^{-\epsilon}W(k)\,d_\epsilon k\int_0^1 \exp[-\tfrac{1}{2}k^2\beta(\eta - \eta^2)]\,d\eta \tag{20}$$

For Gaussian white noise $W(k) = \gamma$ and the variation principle leads to the one-dimensional bound

$$\langle Z\rangle/L > (2\pi\beta)^{-1/2}\exp[\tfrac{1}{2}(\tfrac{1}{2}\pi)^{1/2}\gamma\beta^{3/2}] \tag{21}$$

This perturbation-theoretic type of bound is superior to the Luttinger form at high temperatures but inferior at low temperatures, where it fails to take into account the fluctuations in the random potential that lead to deep traps. One can proceed to a more systematic theory by using a cumulant expansion. The result of calculating the second cumulant is a term in the exponent of order β^3, leading to small modifications at high temperatures, but exceeding the first term at low temperatures, so that the cumulant series is then useless. We lose the variational bound, but do generate the perturbation expansion at high temperatures.[14]

The calculation of the terms of the perturbation expansion is straightforward. We have

$$\langle A^n \rangle = (1/2^n)(2\pi)^{-\epsilon n} \iint W(k_1) W(k_n)\, d_\epsilon k_1 \cdots d_\epsilon k_n$$
$$\times \iiint dt_1\, ds_1 \cdots dt_n\, ds_n \left\langle \exp \int f(u) r(u)\, du \right\rangle$$

with

$$f(u) = i \sum_{i=1}^{n} k_i[\delta(u - t_i) - \delta(u - s_i)] \tag{22}$$

The functional average leads to a quadratic form in the k_i. Dimensional analysis leads immediately to the β^3 result for $\langle A^2 \rangle - \langle A \rangle^2$.

The next idea is to choose an S_0 that represents a particle moving in a potential localized about r_0 (the starting point of a path). This idea has been developed by Friedberg and Luttinger for the Edwards–Gulyaev functional. The potential will be temperature dependent, so that at high temperatures it can be weakened to yield the perturbation result. However, there is an important difficulty inherent in this approach, which we now discuss. We have

$$S_0 = \tfrac{1}{2} \int_0^\beta \dot{r}^2(u)\, du + \int_0^\beta V(r(u))\, du \tag{23}$$

In the Feynman approximation we must evaluate $\int e^{-S_0}\, \mathscr{D}r(u)$, $\langle A \rangle_{S_0}$, and $\langle \int_0^\beta V(r(u))\, du \rangle_{S_0}$. The last two quantities involve functional integrals for a particle in a potential with a superimposed uniform electric field that acts impulsively. It is not easy to handle over the entire temperature range. However, the main features of the theory can be ascertained by using a trial potential of the harmonic oscillator form

$$V(r) = \tfrac{1}{2}\omega^2 (r - r_0)^2 \tag{24}$$

The path integral form immediately gives the factor L^ϵ and we can take $r_0 = 0$. Using well-known formulas for the oscillator,[12] we have for white noise in one dimension

$$\langle Z \rangle / L > (\omega / 2\pi \sinh \omega\beta)^{1/2} \exp\langle A \rangle \exp\langle B \rangle \tag{25}$$

$$\langle B \rangle \equiv \tfrac{1}{2}\omega^2 \left\langle \int_0^\beta r^2(u)\, du \right\rangle = \tfrac{1}{4}\omega\beta \coth(\omega\beta) - \tfrac{1}{4} \tag{26}$$

We will compute $\langle A \rangle$ shortly. At low temperatures it takes the limiting form

$$\langle A \rangle \to \tfrac{1}{2}\gamma\beta^2 (\omega/2\pi)^{1/2}, \qquad \omega\beta \to \infty \tag{27}$$

The frequency $\omega(\beta)$ can be chosen to optimize the bound. In the limit of low temperatures we find

$$\omega = (\beta^2/2\pi)\gamma^2 \tag{28}$$

$$\langle Z\rangle/L \to (\omega/\pi)^{1/2} \exp(\beta^3\gamma^2/8\pi) \tag{29}$$

The prefactor differs from the Luttinger result and leads to a density of states proportional to $|E|^{1/4}$, which is better, but not correct (cf. Section 5).

To calculate $\langle A\rangle$ over the entire temperature range we use results from the forced harmonic oscillator

$$\left\langle \exp \int_0^\beta f(u)r(u)\,du \right\rangle_{\mathrm{HO}} = \exp\left[\tfrac{1}{2}\int_0^\beta \xi(u)f(u)\,du\right] \tag{30}$$

with

$$\int_0^\beta f(u)\xi(u)\,du = -\int_0^\beta f(u)\,du\,\frac{G(u)}{\omega} + \frac{G(\beta)}{\omega\sinh\omega\beta}\int_0^\beta \sinh(\omega u)\,f(u)\,du$$

$$G(u) = \int_0^\beta \sinh[\omega(u-u')]\,f(u')\,du' \tag{31}$$

With $f(u) = ik[\delta(u-t) - \delta(u-s)]$, we have

$$\langle A\rangle = \tfrac{1}{2}\beta^2(\omega/2\pi)^{1/2}\int_0^1\int_0^1 dt\,ds\,[F(t,s)]^{-1/2} \tag{32}$$

$$F(t,s) = \sinh[\omega\beta(t-s)] + 4\sinh^2\left(\omega\beta\frac{s-t}{2}\frac{\cosh[\omega\beta(t+\frac{1}{2}s)]\cosh\{\omega\beta[1-(t+\frac{1}{2}s)]\}}{\sinh\omega\beta}\right) \tag{33}$$

The low-temperature limit leads to the results quoted earlier. At high temperatures ($\omega\beta \ll 1$) we find the expansion

$$\langle A\rangle \to \frac{\gamma\beta^{3/2}}{2}\left(\frac{\pi}{2}\right)^{1/2}\left[1 - \frac{(\omega\beta)^2}{12}\left(\frac{3\pi}{4} - \frac{4}{3}\right) + \cdots\right] \tag{34}$$

with a negative coefficient for the $(\omega\beta)^2$ term. In the same region

$$\begin{aligned}\langle B\rangle &\to \frac{(\omega\beta)^2}{12} + O[(\omega\beta)^4] \\ \left(\frac{\omega\beta}{\sinh\omega\beta}\right)^{1/2} &= \exp\left(-\frac{1}{2}\ln\frac{\sinh\omega\beta}{\omega\beta}\right) \to \exp\left[-\frac{(\omega\beta)^2}{12} + \cdots\right]\end{aligned} \tag{35}$$

Thus the high-temperature limit of the partition function is

$$\begin{aligned}\langle Z\rangle/L \to\; & (2\pi\beta)^{-1/2}\exp[\tfrac{1}{2}\beta^{3/2}\gamma(\tfrac{1}{2}\pi)^{1/2}] \\ & \times \exp[-(\omega\beta)^2\gamma\beta^{3/2}(\tfrac{1}{2}\pi)^{1/2}(1/24)(\tfrac{3}{4}\pi - \tfrac{4}{3}\pi)]\end{aligned} \tag{36}$$

In the high-temperature limit any nonzero choice of the oscillator frequency ω leads to a worse result than the perturbation choice $\omega = 0$. We thus have the following conclusion. At low temperatures one can choose the oscillator frequency to obtain the exponential low-energy tail (with the oscillator coefficient 8π). There will be a critical temperature above which one must shift to zero frequency and use the perturbation cumulant expansion. As a result, some derivative of the partition function will be discontinuous. This behavior is entirely analogous to that in the polaron problem. A switch from a delocalized to a localized potential occurs at a critical coupling constant. While we know of no proof that some type of discontinuity cannot occur, it is here clearly an artifact of the approximation under discussion. The Feynman two-time quadratic trial action yields a smooth transition and a lower energy at all coupling constants. The main defect of the bound oscillator model is the failure to respect the translational invariance of the original action fully. This is seen in polaron theory by noting the form of the canonical transformations that yield results identical to those based on a localized single-time trial action with potential centered about the starting point.(16,17) If there are discontinuities in $Z(\beta)$ or its temperature derivatives, it would require far more refined considerations than any heretofore given to demonstrate that fact.

One can of course use a potential that gives more accurate results than the oscillator potential in the low-temperature limit, just as in our discussion of the Luttinger principle. However, the resulting theory is expected to have the same defect as one goes to the high-temperature limit. It is interesting that even at low temperatures the prefactor has an incorrect β dependence. This leads to an incorrect energy dependence of the prefactor in the density of states. The theory based on a two-time quadratic trial action corrects this. In addition, surprisingly but gratifyingly, the formulas are simpler than for the case of bound potential. Hence we do not pursue the discussion of a systematic cumulant expansion. This has been analyzed by Friedberg and Luttinger(9) for the more difficult Edwards–Gulyaev action, with special emphasis on the low-temperature limit.

3. FEYNMAN APPROXIMATION WITH A TWO-TIME ACTION

The present work is based on a quadratic two-time action

$$S_0 = \frac{1}{2}\int_0^\beta \dot{r}^2\, du + \frac{\omega^2}{4\beta}\int_0^\beta\int_0^\beta du\, du'\, [r(u) - r(u')]^2 \tag{37}$$

This has the same translational invariance as S. We may write

$$S_0 = S_{\mathrm{HO}} - (\omega^2/2\beta)\left[\int_0^\beta r(u)\,du\right]^2$$
$$S_{\mathrm{HO}} = \tfrac{1}{2}\int_0^\beta \dot{r}^2(u)\,du + \tfrac{1}{2}\omega^2\int_0^\beta r^2(u)\,du \tag{38}$$

Our starting point is a simple formula for the path integral $\int_{r_0} e^{-S_0}\,\mathscr{D}r(u)$. Because of the translation-invariant form of S_0, we may take $r_0 = 0$. We use the parameteric representation

$$\exp\left\{(\omega^2/2\beta)\left[\int_0^\beta r(u)\,du\right]^2\right\}$$
$$= (2\pi)^{-\epsilon/2}\int_{-\infty}^{+\infty} d_\epsilon y\,\exp(-y^2/2)\exp\left[(\omega y/\sqrt{\beta})\int_0^\beta r(u)\,du\right] \tag{39}$$

Then

$$\int (\exp - S_0)\,\mathscr{D}r(u)$$
$$= \int (\exp - S_{\mathrm{HO}})\,\mathscr{D}r(u)(2\pi)^{-\epsilon/2}$$
$$\times \int_{-\infty}^{+\infty} d_\epsilon y\,[\exp(-y^2/2)]\left\langle \exp\left[(\omega y/\sqrt{\beta})\int_0^\beta r(u)\,du\right]\right\rangle_{\mathrm{HO}} \tag{40}$$

The theory of the forced harmonic oscillator leads to

$$\int (\exp - S_{\mathrm{HO}})\,\mathscr{D}r(u) = (\omega/2\pi\sinh\omega\beta)^{\epsilon/2} \tag{41}$$

$$\left\langle \exp\left[\frac{\omega y}{\sqrt{\beta}}\int_0^\beta r(u)\,du\right]\right\rangle_{\mathrm{HO}} = \exp\left\{\frac{y^2}{2}\left[1 - \frac{\tanh(\omega\beta/2)}{\omega\beta/2}\right]\right\} \tag{42}$$

This yields the fundamental result

$$\int_{r_0} e^{-S_0}\,\mathscr{D}r(u) = \left[\frac{1}{(2\pi\beta)^{1/2}}\,\frac{\omega\beta/2}{\sinh(\omega\beta/2)}\right]^\epsilon \tag{43}$$

We will see that this factor already yields a different (and correct) prefactor for the density of states than for the theories discussed earlier. Let us write

$$S_0 - S = A + B$$
$$A = \int_0^\beta\int_0^\beta W(r(u) - r(u'))\,du\,du' \tag{44}$$
$$B = (\omega^2/4\beta)\int_0^\beta\int_0^\beta du\,du'\,[r(u) - r(u')]^2$$

The variational principle leads to

$$\frac{Z}{L} \geqslant \left(\frac{1}{(2\pi\beta)^{1/2}}\,\frac{\omega\beta/2}{\sinh(\omega\beta/2)}\right)^\epsilon e^{\langle A\rangle}e^{\langle B\rangle} \tag{45}$$

where

$$\langle A \rangle = \left(\int e^{-S_0} \mathscr{D}r \right)^{-1} \int Ae^{-S_0} \mathscr{D}r \tag{46}$$

The average of B with the action S_0 is most easily evaluated by making the replacement $\omega^2 \to \omega^2\lambda$. Then

$$\langle B \rangle = -(\partial/\partial\lambda)\left[\ln \int e^{-S_0} \mathscr{D}r(u) \right] \tag{47}$$

evaluated at $\lambda = 1$. We find

$$\frac{\langle B \rangle}{\epsilon} = -\frac{1}{2\lambda} + \frac{\omega\beta}{4} \frac{1}{\sqrt{\lambda}} \coth \frac{\omega\beta\sqrt{\lambda}}{2} \tag{48}$$

To calculate $\langle A \rangle_{S_0}$, we write

$$\begin{aligned}\langle A \rangle = \tfrac{1}{2}(1/2\pi)^{\epsilon} \int_{-\infty}^{+\infty} W(k)\, d_{\epsilon}(k) \\ \times \int_0^{\beta} \int \left\langle \exp \int_0^{\beta} f(u) r(u)\, du \right\rangle_{S_0} dt\, ds\end{aligned} \tag{49}$$

with $f(u) = ik[\delta(u - t) - \delta(u - s)]$. The formula for the average contained in $\langle A \rangle$ follows from the standard analysis of Gaussian functional integrals. It is

$$\left\langle \exp \int_0^{\beta} f(u) r(u)\, du \right\rangle_{S_0} = \exp\left[\tfrac{1}{2} \int_0^{\beta} \xi(u) f(u)\, du \right] \tag{50}$$

where $\xi(u)$ is a solution of

$$\ddot{\xi} - (\omega^2/\beta) \int_0^{\beta} [\xi(u) - \xi(u')]\, du' = -f(u), \qquad \xi(0) = \xi(\beta) = 0 \tag{51}$$

To exhibit the solution we define

$$\begin{aligned} G(u) &= \int_0^{u} \{\sinh[\omega(u - u')]\} f(u')\, du' \\ C &= \tfrac{1}{2}\omega \int_0^{\beta} G(u)\, du - [\cosh(\tfrac{1}{2}\omega\beta)] G(\beta)/2 \\ D &= -\tfrac{1}{2}\omega[\cosh(\tfrac{1}{2}\omega\beta)] \int_0^{\beta} G(u)\, du + \tfrac{1}{2}G(\beta) \end{aligned} \tag{52}$$

The solution is then

$$-\omega\xi(u) = G(u) + C \sinh \omega u + D(\cosh \omega u - 1) \tag{53}$$

With $f(u)$ as noted above

$$\begin{aligned}&-\omega \int_0^{\beta} \xi(u) f(u)\, du \\ &\quad = +k^2\{+\sinh[\omega(t - s)] + [\cosh(\tfrac{1}{2}\omega\beta)][1 - \cosh \omega(t - s)]\}\end{aligned} \tag{54}$$

Note that this result depends only on the time difference $t - s$, in contrast to the bound oscillator formula, which also involves $t + s$. We now measure t and s in units of β and introduce

$$\Delta(\xi) = \tanh(\tfrac{1}{2}\omega\beta)\sinh(\omega\beta\xi) - \cosh(\omega\beta\xi) \tag{55}$$

Then

$$\langle A\rangle_S = \frac{\beta^2}{2(2\pi)^\epsilon}\left(\omega\tanh\frac{\omega\beta}{2}\right)^{\epsilon/2}\int_\infty^\infty d_\epsilon k\, W\left(k\left(\omega\tanh\frac{\omega\beta}{2}\right)^{1/2}\right)$$
$$\times\int_0^1 \exp -\frac{k^2}{2[1+\Delta(\xi)]}\, d\xi \tag{56}$$

Let us now specialize to the case of one-dimensional white noise. Then

$$\langle A\rangle_{S_0} = \tfrac{1}{2}\beta^2(\omega/2\pi)^{1/2}\gamma\Gamma(\omega\beta) \tag{57}$$

with

$$\Gamma(\omega\beta) = \tanh(\omega\beta/2)^{1/2}\int_0^1 [1+\Delta(\xi)]^{-1/2}\, d\xi$$

The quantity Γ can be reduced to a known integral by elementary transformations. Let

$$\mu \equiv \{2/[1+\coth(\tfrac{1}{2}\omega\beta)]\}^{1/2} \tag{58}$$

Then,

$$\Gamma = (2/\omega\beta)\mu K(\mu) \tag{59}$$

where $K(\mu)$ is the complete elliptic integral.[8] Our final answer for the one-dimensional white noise case is

$$\frac{Z}{L} \geqslant \frac{1}{(2\pi\beta)^{1/2}}\,\frac{\omega\beta/2}{\sinh(\omega\beta/2)}\exp\left(\frac{\omega\beta}{4}\coth\frac{\omega\beta}{2} - \frac{1}{2}\right)\exp\frac{\gamma\beta^2\omega^{1/2}\Gamma(\omega\beta)}{2(2\pi)^{1/2}} \tag{60}$$

4. LOW- AND HIGH-TEMPERATURE LIMITS

In this section we examine the low- and high-temperature limits of the formula derived at the end of the last section.

We first examine the low-temperature limit. The behavior of the elliptic integral for μ near unity is

$$K(\mu) = \ln(4e^{\omega\beta/2}) + O(\omega\beta e^{-\omega\beta}) \tag{61}$$

Then

$$\Gamma \to 1 + [(2\ln 4)/\omega\beta] + O(e^{-\omega\beta}) \tag{62}$$

The corrections involve exponentials in $\omega\beta$. We find $\langle Z\rangle/L$ in the approximation that all exponentials $e^{-\omega\beta}$ in the exponential are to be discarded:

$$\frac{\langle Z\rangle}{L} \to \frac{\omega\beta}{(2\pi\beta)^{1/2}} \exp\left(-\frac{\omega\beta}{4} + \frac{\gamma\beta^2\sqrt{\omega}}{2(2\pi)^{1/2}}\right) \exp \frac{\gamma\beta^2\sqrt{\omega}}{(2\pi)^{1/2}} \frac{\ln 4}{\omega\beta} \exp -\frac{1}{2} \tag{63}$$

For $\omega\beta \gg 1$ we can choose $\omega(\beta)$ to maximize the first exponential. This leads again to $\omega = \beta^2\gamma^2/2\pi$ and to a partition function

$$\frac{\langle Z\rangle}{L} \to \frac{1}{(2\pi\beta)^{1/2}} \gamma^2 \frac{\beta^3}{2\pi} 4 \exp -\frac{1}{2} \exp \frac{\gamma^2\beta^3}{8\pi} \tag{64}$$

The dominant exponent is the same as the Luttinger value with an oscillator. However, the prefactor now has a different β dependence and leads to the correct $|E|$ dependence of the density of states in the low-energy tail. The absolute value of the coefficient of $|E|$ differs from the exact result by 36%.

In the high-temperature region, we wish to determine whether there are choices of $\omega(\beta)$ that are nonzero and lead to a superior bound to the $\omega = 0$ case. We first examine the expansions of all terms other than Γ. For $\omega\beta < 1$,

$$\langle B\rangle \to \frac{(\omega\beta)^2}{24} - \left(\frac{\omega\beta}{2}\right)^4 \frac{1}{90} + \ln \frac{\sinh(\omega\beta/2)}{\omega\beta/2} \to \left(\frac{\omega\beta}{2}\right)^2 \frac{1}{3!} - \left(\frac{\omega\beta}{2}\right)^4 \frac{1}{80} \tag{65}$$

Thus

$$\frac{\langle Z\rangle}{L} \to (2\pi\beta)^{-1/2} \exp\left[-\left(\frac{\omega\beta}{Z}\right)^4 \frac{1}{180}\right] \exp\left(\frac{\gamma\beta^2\sqrt{\omega}}{2(2\pi)^{1/2}}\,\Gamma\right) \tag{66}$$

Using the expansion

$$\Gamma = \frac{\pi}{\omega\beta}\,\omega\left[1 + \frac{1}{4}\mu + \left(\frac{3}{\gamma}\right)^2 \mu^2 + \cdots\right] = \frac{\pi}{(\omega\beta)^{1/2}}\left[1 + \frac{(\omega\beta)^{1/2}}{4} + O(\omega\beta) + \cdots\right] \tag{67}$$

we have

$$\frac{\langle Z\rangle}{L} \to \frac{1}{(2\pi\beta)^{1/2}} \exp\left[\frac{\gamma\beta^{3/2}}{2}\left(\frac{\pi}{2}\right)^{1/2}\right] \exp(F) \tag{68}$$

The first two factors yield the zero-frequency estimate. Here

$$F = \frac{\gamma\beta^{3/2}}{2}\left(\frac{\pi}{2}\right)^{1/2} \frac{(\omega\beta)^{1/2}}{4} - \left(\frac{\omega\beta}{2}\right)^4 \frac{1}{180} \tag{69}$$

It is made a maximum with

$$\omega\beta = [45\gamma(\pi/2)^{1/2}]^{2/7}\beta^{3/7} \tag{70}$$

F is positive with the second term equal to 1/8 of the first term.

We note that $\omega\beta \to 0$ weakly as $\beta \to 0$, so that the expansion is consistent. But ω itself increases as $\beta \to 0$ as $\beta^{-4/7}$. Thus the optimum $\omega(\beta)$ has a minimum at some intermediate value of β. Note that the dependence of Z on γ is not that of higher order perturbation theory. That theory depends on letting $\gamma \to 0$ before letting $\beta \to 0$. The preceding calculation reverses that order.

We have the freedom to choose $\omega(\beta)$ in a less than optimal way in order to have a tractable function $Z(\beta)$. If we take the low-temperature form $\omega = \beta^2\gamma^2/2\pi$ to hold at all temperatures, $Z(\beta)$ becomes a function of the single variable $(\beta^3\gamma^2)$, apart from the factor $(2\pi\beta)^{-1/2}$. Furthermore, the bound improves the $\omega = 0$ result at high temperatures, i.e., includes part of the next order perturbation correction. Thus we have a simple expression, which makes a smooth transition between low and high temperatures, including the next to leading terms at both ends. However, it is not clear how accurate such a theory is in the intermediate domain.

5. DENSITY OF STATES IN THE LOW-ENERGY TAIL

Since the density of states has a low-energy tail that extends to $E \to -\infty$, we use the two-sided Laplace transform

$$\begin{aligned} \frac{Z(\beta)}{L^{\epsilon}} &= \int_{-\infty}^{+\infty} e^{-\beta E} n(E)\, dE \\ n(E) &= \frac{1}{2\pi i}\int_{Ci-\infty}^{C+i\infty} \frac{Z(\beta)}{L^{\epsilon}}\, e^{\beta E}, \qquad c > 0 \end{aligned} \tag{71}$$

We have found $Z(\beta)$ for real, positive β, and the expression represents the analytic continuation for complex β with a positive, real part. To study low-energy tails we must evaluate

$$n(E) = (1/2\pi i)Q_b \int_{C-i\infty}^{C+i\infty} \beta^b \exp(-\beta|E|)\exp(\beta^3/3\alpha^2)\, d\beta \tag{72}$$

Here $\alpha^2 = 8/\gamma^2$ for the correct asymptotic limit and $8\pi/3\gamma^2$ in the oscillator approximation. b and Q_b vary according to the model. The exponent $h(\beta) = \beta^3/3\alpha^2 - \beta|E|$ has a minimum on the positive, real axis at $\beta_0 = \alpha|E|^{1/2}$.

We perform the integral along a line parallel to the imaginary axis, passing through β_0. Then

$$\begin{aligned} n(E) = {} & (1/2\pi)Q_b\alpha^{b+1/2}|E|^{(b/2-1/4)}\pi^{1/2} \\ & \times [\exp(-(2/3)\alpha|E|^{3/2})]J \end{aligned} \tag{73}$$

where

$$J = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dz\,(\exp -z^2)\left(\exp \frac{-iz^3}{3\alpha^{1/2}|E|^{3/4}}\right)\left(1 + i\frac{z\alpha}{\beta_0^{3/4}}\right)^b \tag{74}$$

The main contribution comes from $z < 1$ as $|E| \to \infty$, so that $J \to 1$.

In the Luttinger approximation

$$b = -\tfrac{1}{2}, \qquad Q_b = (2\pi)^{-1/2}, \qquad \alpha^2 = 8/\gamma^2 \tag{75}$$

In the bound oscillator approximation

$$b = 1, \qquad Q_b = \gamma(2\pi)^{-1/2}, \qquad \alpha^2 = 8\pi/3\gamma^2 \tag{76}$$

The variational theory with the two-time quadratic action has

$$b = 5/2, \qquad Q_b = 4\gamma^2/(2\pi)^{3/2}, \qquad \alpha^2 = 8\pi/3\gamma^2 \tag{77}$$

It leads to the expression

$$n(E) \to \frac{1}{\gamma}\,|E|\,\frac{4}{\pi}\,4\pi^{1/2}3^{-3/2}\exp\left[-\frac{2}{3}\left(\frac{8\pi}{3}\right)^{1/2}\frac{|E|^{3/2}}{\gamma}\right] \tag{78}$$

We will see in the next section that consideration of the higher order cumulants drives α^2 to the exact value and changes Q_b. It leaves the index b, which is already exact, unaltered. At the present state the coefficient in Eq. (78) is to be compared with the exact $4/\pi$, so that there is a 36% error.

6. CUMULANT DEVELOPMENT

The second cumulant is

$$\begin{aligned} C_2 &= \langle (A + B)^2\rangle - \langle A + B\rangle^2 \\ &= \langle A^2\rangle - \langle A\rangle^2 + \langle B^2\rangle - \langle B\rangle^2 + 2[\langle AB\rangle - \langle A\rangle\langle B\rangle] \end{aligned} \tag{79}$$

and the contribution to the exponent is $(1/2!)C_2$.

The easiest quantity to evaluate is

$$\begin{aligned} \langle B^2\rangle - \langle B\rangle^2 &= -\frac{\partial}{\partial\lambda}\langle B\rangle \\ &= \epsilon\left[\frac{\omega\beta}{8}\frac{1}{\lambda^{3/2}}\coth\left(\frac{\omega\beta\sqrt{\lambda}}{2}\right) + \frac{(\omega\beta)^2}{16}\operatorname{csch}^2\left(\frac{\omega\beta\sqrt{\lambda}}{2}\right) - \frac{1}{2\lambda^2}\right] \end{aligned} \tag{80}$$

evaluated at $\lambda = 1$.

To evaluate the $\langle AB\rangle$ term, replace ω by $\omega\sqrt{\lambda}$ in $\langle A\rangle$. Then

$$\langle AB\rangle - \langle A\rangle\langle B\rangle = -(\partial/\partial\lambda)\langle A\rangle \tag{81}$$

In the one-dimensional white noise problem at low temperatures

$$\Gamma(\beta\omega\sqrt{\lambda}) \to 1 + [(2\ln 4)/\omega\beta\sqrt{\lambda}] + \text{exponential terms} \tag{82}$$

$$\begin{aligned} \langle AB\rangle - \langle A\rangle\langle B\rangle &= -(\partial/\partial\lambda)[\tfrac{1}{2}\beta^2(\omega/2\pi)^{1/2}\lambda^{1/4}\Gamma(\beta\omega\sqrt{\lambda})]\gamma \\ &\to -\tfrac{1}{8}\beta^2(\omega/2\pi)^{1/2}\gamma \end{aligned} \tag{83}$$

The expression for $\langle A^2 \rangle$ follows from the earlier type of consideration:

$$\langle A^2 \rangle = \tfrac{1}{4}(2\pi)^{-\epsilon} \iint d_\epsilon k_1\, d_\epsilon k_2\, W(k_1)W(k_2) \times \int_0^\beta \cdots \int_0^\beta dt_1 \cdots ds_2 \exp\left[\tfrac{1}{2}\int_0^\beta \xi(u)f(u)\,du\right] \tag{84}$$

with

$$f(u) = ik_1[\delta(u - t_1) - \delta(u - s_1)] + ik_2[\delta(u - t_2) - \delta(u - s_2)]$$

Then,

$$\tfrac{1}{2}\int_0^\beta f(u)\xi(u)\,du = -(1/2\omega)[\tanh(\tfrac{1}{2}\omega\beta)]\{k_1{}^2[1 + D(t_1 - s_1)] + k_2{}^2[1 + D(t_2 - s_2)] - k_1k_2D_{12}\} \tag{85}$$

$$\begin{aligned} D(|t_1 - s_1|) &= [\tanh(\tfrac{1}{2}\omega\beta)]\sinh(\omega|t_1 - s_1|) - \cosh(\omega|t_1 - s_1|) \\ D_{12} &= D(|t_1 - t_2|) + D(|s_1 - s_2|) - D(t_1 - s_2|) - D(|t_2 - s_1|) \end{aligned} \tag{86}$$

For one-dimensional Gaussian white noise

$$\langle A^2 \rangle = \frac{\gamma^2}{4\cdot 2\pi}\beta^4\omega\left(\tanh\frac{\omega\beta}{2}\right)\int_0^1\int dt_1\, ds_1\, dt_2\, ds_2 \times \left\{[1 + \Delta(t_1 - s_1)][1 + \Delta(t_2 - s_2)] - \frac{\Delta_{12}^2}{4}\right\}^{-1/2} \tag{87}$$

The fourfold integration actually depends on three independent differences, but is still very complicated. It is, however, easy to find results sufficiently accurate to treat the low-temperature limit. We examine the expansion in powers of

$$y = \Delta_{12}^2/[1 + \Delta(t_1 - s_1)][1 + \Delta(t_2 - s_2)] \tag{88}$$

We have

$$\langle A^2 \rangle - \langle A \rangle^2 = \frac{\gamma^2}{4\cdot 2\pi}\beta^4\omega\left(\tanh\frac{\omega\beta}{2}\right) \times \iiiint \frac{dt_1\, ds_1\, dt_2\, ds_2}{[1 + \Delta(t_1 - s_1)]^{1/2}[1 + \Delta(t_2 - s_2)]^{1/2}} \times \left(\frac{y}{8} + \frac{3}{8}\frac{y^2}{4^2} + \cdots\right) \tag{89}$$

The key point is that the leading term as $\omega\beta \to \infty$ comes from terms with only one time difference, e.g., $\Delta^2(t_1 - t_2)$, $\Delta^4(t_1 - t_2)$, etc. Thus we may neglect all of the denominators. The first power of y contributes four identical terms, as do all the higher powers. Thus we have

$$\langle A^2\rangle - \langle A\rangle^2 \to \frac{\gamma^2\beta^4\omega}{4\cdot 2\pi}\left\{\frac{1}{2}4\frac{\int_0^1 \Delta^2(\xi)\,d\xi}{4} + \frac{3}{8}4\frac{\int_0^1 \Delta^4\,d\xi}{4^2} + \cdots\right\} \tag{90}$$

asymptotically

$$\int_0^1 \Delta^2(\xi) = 1/\beta\omega, \qquad \int_0^1 \Delta^4\,d\xi = 1/(2\beta\omega) \tag{91}$$

The leading term in $\langle A^2\rangle - \langle A\rangle^2$ cancels the other two terms in the second cumulant. Thus the contribution to the exponent is

$$\frac{1}{2}C_2 = \frac{\gamma^2\beta^3}{8\pi}\left(\frac{3}{64} + \cdots\right) \tag{92}$$

The correction to the Feynman approximation is $(1/\pi)[1 + (3/64) + \cdots]$, which is almost the exact result of 1/3. Thus the main effect of C_2 is to correct the coefficient of the dominant term. The energy dependence of the prefactor is unaffected since it corresponds to terms of order $\ln(\omega\beta)$ in an exponential and no such terms occur in the expansion of C_2.

To complete the discussion, we estimate the effect of the second cumulant correction on the numerical value of the prefactor. This arises from the non-exponential terms in the second cumulant. There is no contribution from $\langle AB\rangle - \langle A\rangle\langle B\rangle$ and that from $\frac{1}{2}\{\langle B^2\rangle - \langle B\rangle^2\}$ is $-\frac{1}{4}$.

The dominant contribution comes from the first term, linear in y (cf. the appendix for the treatment of the integrals). We find, accurate to order $(\omega\beta)^{-2}$,

$$\begin{aligned}\int \Delta_{12}^2(1 + \Delta_1)^{-3/2}(1 + \Delta_2)^{-3/2}\,dt_1\,ds_1\,dt_2\,ds_2 \\ = \frac{4}{\omega\beta} + \frac{16}{\omega^2\beta^2}(1 - 2\ln 2)\end{aligned} \tag{93}$$

and

$$\frac{\langle A^2\rangle - \langle A\rangle^2}{2} = \frac{\beta^3\gamma^2}{32\pi} + \frac{\beta^2\gamma^2}{\omega}\frac{1}{8\pi}(1 - 2\ln 2) \tag{94}$$

The numerical value of the prefactor becomes $(4/\pi)(\pi/3)^{1/2}(4/3)(1/\sqrt{2})$, since the contribution of the second cumulant is a factor $\frac{1}{2}\sqrt{2}$. This is too low by 3%.

The term in y^2 can be computed, using

$$\int \Delta_{12}^4\,dt_1\,ds_1\,dt_2\,ds_2 = \frac{2}{\omega\beta} + \frac{44}{3}\frac{1}{\omega^2\beta^2} \tag{95}$$

It contributes an additional factor $\exp[11/(16)^2]$. The total result is a prefactor accurate to 1%.

We note that in the high-temperature region, the $\omega = 0$ limit already includes all terms of order $\beta^3\gamma^2$ and a suitable $\omega(\beta)$ improves the accuracy.

The third-order cumulant contributes $(1/3!)C_3$ to the exponential

$$C_3 = \langle (A + B)^3 \rangle - \langle A + B \rangle^3 - 3\langle A + B \rangle(\langle A + B \rangle^2 - \langle A + B \rangle^2) \quad (96)$$

All of the terms except for $\langle A^3 \rangle - \langle A \rangle^3$ may be computed from cited formulas by replacing ω by $\omega\sqrt{\lambda}$ and differentiating with respect to λ. The structure of $\langle A^3 \rangle$ is similar to (10) and is easily written down. We have not calculated the corrections in the low-temperature limit, since they appear to be very small.

7. DISCUSSION

The quantum mechanical theory of an electron subjected to Gaussian noise is the simplest example of a set of problems covering a wide range of phenomena. More complicated cases include the behavior of an electron interacting with randomly placed scatterers or with lattice vibrations and certain polymer configuration problems. The order of difficulty is transparent in the path integral formalism, where one inspects the action functional. All of these problems have to cope with deep traps or bound states in a manner that does justice to the overall translational invariance. In addition, one wants an accurate representation of the higher energy parts of the spectrum where the traps or bound states are not important.

In most of these cases there exists the outlines of a profound systematic treatment of the low-energy states. For the Gaussian noise problem it is the theories of Halperin and Lax, Zittarz and Langer, and Edwards and Freed. Methods were developed to deal with isolated deep traps, with additional small fluctuations, in a manifestly translation-invariant form. The theory computes not only the dominant exponential low-energy tail in the density of states, but also the energy dependence and constants in the "prefactors." These theories also hold in three dimensions and for nonwhite noise. They can be checked for accuracy by comparing with the exact one-dimensional results.

For the polaron there is no exact theory even in one dimension. The corresponding theory is the strong coupling adiabatic theory of Bogolyubov and Tyablikov.[19] This approach was developed in Hamiltonian form with auxiliary variables a long time ago. Recently it has been revived, expressed in somewhat varying forms, and studied more deeply.[20,21] The same basic ideas have been examined in the relativistic quantum field theory of "extended" objects, which has been studied from many points of view.[22] This includes work within the path integral formalism.[23] This theory gives an account of the low-lying excitation structure and of their zero-point energy contributions to the ground state. For example, in the partition function and density of states of the polaron, they become a theory of the "prefactor." The dominant exponential is already given by a primitive, symmetry-breaking Hartree approximation. However, there has been no successful linkage of the strong coupling and weak coupling limits, other than variational interpolation schemes for a few quantities such as the ground-state energy and effective mass. These schemes, which use conventional Hamiltonian methods, do not allow us to see how the rich structure of the strong coupling limit disappears as one goes to weaker coupling strengths.

The most successful overall theory of the polaron is the path integral treatment of Feynman based on a two-time quadratic trial action. It has been applied to a host of problems, including electrical, optical, magnetic, and thermal properties of polarons and excitons.[24] It gives satisfactory results for these problems over the entire range of coupling constants.

There are, however, some things that are left out of the Feynman treatment. In particular, it does not describe the finer details of the systematic adiabatic theory in the strong coupling limit. We do not refer to the limitations of the oscillator approximation in estimating the dominant term in the ground-state energy. This is a relatively minor matter and is certainly remedied in a cumulant extension of the theory. More important is the need to improve the Feynman approach to describe the low-lying excited states, including the bound photon polaron states. Work has been done with quadratic actions with a general explicit time delay, rather than the simple exponential delay of the original theory. But the connection with the strong coupling theory is not at all clear and merits further study.

After observing that the Zittarz–Langer theory of Gaussian noise is on a par with strong coupling polaron theory, we thought it worthwhile to examine the associated two-time quadratic action approximation. Since the problem is easier than the polaron problem, one can obtain explicit results and drive the calculations to higher orders, using the cumulant expansion. We have verified that one does indeed pick up the finer features of the deep trap theory. This is satisfying, since the path integral treatment also describes the high-temperature limit of the partition function in a very satisfactory manner. However, the treatment does not really incorporate the ideas of the deep trap theory. This merits further thought. We would expect to find the same sort of thing if the polaron were treated with a cumulant development starting from the Feynman approximation.

It should be noted that the type of trial action used here has been discussed by Abram and Edwards[25] and Bezak[26] for random impurity problems.[3] However, the action that they used resulted from a long-wavelength

[3] See Note Added in Proof on page 286.

expansion of the original action. If one does this for the polaron problem, one does not obtain the correct strong coupling limit. Likewise here one cannot describe the effects of deep traps properly. This type of theory only yields the factor called $\int e^{-S_0}\,\mathscr{D}r(u)$ in the present discussion. It is essential to use the variational principle that includes the contributions of terms $\langle A\rangle$ and $\langle B\rangle$ earlier in this paper.

We hope to apply the results of the present work to more difficult problems involving random potentials similar to those of polaron theory mentioned above, which have been analyzed with Feynman's approach.

APPENDIX

We describe techniques for the evaluation of the integrals occurring in the cumulants. They are particularly useful in the limit of low temperatures, where one neglects exponentials $e^{-\omega\beta}$.

The simplest integral is

$$\int_0^1\int_0^1 \Delta(t_1 - s_1)\,dt_1\,ds_1 = -\frac{1}{(1-e^{-\omega\beta})}\int_0^1 (e^{-\omega\beta\xi} + e^{\omega\beta(1-\xi)})\,d\xi \to -\frac{2}{\omega\beta} \tag{A.1}$$

The integral is symbolized by a line between the points t_1 and s_1,

$$t_1 \text{——} s_1$$

We also have

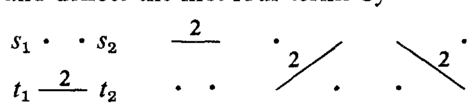
$$\int_0^1\int_0^1 \Delta^m(t_1 - s_1)\,dt_1\,ds_1 \to (2/m\omega\beta)(-1)^m \tag{A.2}$$

Increasing the number of lines between two points does not change the power of $\omega\beta$.

Consider next the integrals occurring in the evaluation of

$$\begin{aligned}\iiiint \Delta_{12}^2\,dt_1\,ds_1\,dt_2\,ds_2\,\Delta_{12}^2 &= \Delta^2(t_1 - t_2) + \Delta^2(s_1 - s_2) + \Delta^2(t_1 - s_2) + \Delta^2(t_2 - s_1)\\ &\quad + 2\Delta(t_1 - t_2)\Delta(s_1 - s_2) + 2\Delta(t_1 - s_2)\Delta(t_2 - s_1)\\ &\quad - 2[\Delta(t_1 - t_2)\Delta(t_1 - s_2) + \Delta(t_1 - t_2)\Delta(t_2 - s_1) + \Delta(s_1 - s_2)\Delta(t_1 - s_2)\\ &\quad + \Delta(s_1 - s_2)\Delta(t_2 - s_1)]\end{aligned} \tag{A.3}$$

Mark four points and denote the first four terms by

(A)  (A.4)

The fourfold integration is symmetric, so all integrals yield the same value. The total contribution is $(4/\omega\beta)$. Next consider the diagrams corresponding to the second line of (A.3):

(B)  (A.5)

These are disconnected and equivalent. Each one contributes $(2/\omega\beta)^2$ and we must supply the factor 2 from the expansion of Δ_{12}^2. So the total contribution is $2 \times 2 \times (2/\omega\beta)^2$.

The diagrams corresponding to the third line are

(C)  (A.6)

We examine the first of the four equivalent diagrams. It corresponds to

$$\iint dt_1\, ds_1 \int_0^1 \Delta(t_1 - t_2)\, dt_2 \int_0^1 \Delta(t_1 - s_2)\, ds_2 \tag{A.7}$$

We note that

$$\int_0^1 \Delta(t_1 - s_2)\, ds_2 = -(2/\omega\beta)(1 - e^{-\omega\beta}) \to -2/\omega\beta \tag{A.8}$$

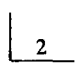
which is independent of t_1. So, any line with a free end may be detached. Supplying the factor (-2) from the expansion of Δ_{12}^2, the total contribution is $(-2)(4)(-2/\omega\beta)^2 = -32/\omega^2\beta^2$.

We then find

$$\iint \Delta_{12}^2\, d\tau = \frac{4}{\omega\beta} - \frac{16}{\omega^2\beta^2} \tag{A.9}$$

where $d\tau \equiv dt_1\, ds_1\, dt_2\, ds_2$.

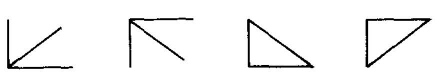
Consider next the computation of $\int \Delta(t_1 - s_1)\, \Delta_{12}^2\, d\tau$. This adds a vertical line on the left of each of the diagrams. Thus the first A diagram becomes

(A′)  (A.10)

The B diagrams become

(B′)  (A.11)

Again the free-ending lines may be detached and each diagram contributes $(-2/\omega\beta)^3$. The C diagrams become

(C′)  (A.12)

The first and second have the same contribution as the B′ diagrams. The two triangle diagrams require a separate calculation. The first one corresponds to

$$E \equiv \iiiint \Delta(t_1 - s_1)\Delta(t_1 - t_2)\Delta(s_1 - t_2)\, d\tau \tag{A.13}$$

The substructure

$$\int_0^1 \Delta(s_1 - t_2)\Delta(t_1 - t_2)\, dt_2$$
$$\to [\exp(-\omega\beta|t_1 - s_1|)]\left(\frac{1}{\omega\beta} + |t_1 - s_1|\right)$$
$$+ \{\exp[\omega\beta(|t_1 - s_1|)] \exp(-\omega\beta)\}\left(1 + \frac{1}{\omega\beta} - |t_1 - s_1|\right) \quad \text{(A.14)}$$

where we have dropped terms of order $e^{-\omega\beta}$. Taking the integral over half the range and doubling the result, we need only compute the first half. Hence

$$E \to -2 \int_0^{1/2} e^{-\omega\beta\xi} e^{-\omega\beta\xi}\left(\frac{1}{\omega\beta} + \xi\right) d\tau \to -(3/2)(\omega^2\beta^2)^{-1} \quad \text{(A.15)}$$

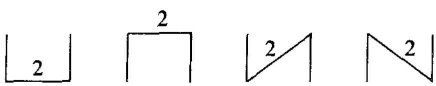
We see that in computing

$$\int \Delta_1{}^n(t_1 - s_1)\Delta(t_1 - t_2)\Delta(s_1 - t_2)\, d\tau$$

the first factor is replaced by $\exp(-n\omega\xi)$. So

$$E_n = -2(\omega\beta)^{-2}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2}\right) \quad \text{(A.16)}$$

We next consider $\iiiint \Delta(t_1 - s_1)\Delta(t_2 - s_2)\Delta_{12}^2\, d\tau$ and show that it is of order $(\omega\beta)^{-3}$. The A diagrams become

(A″)  (A.17)

The extra free index line factors and each diagram is of order $(\omega\beta)^{-3}$. We are neglecting terms of this order. The B diagrams become

(B″)  (A.18)

It is easily shown that both of these, which are connected diagrams involving all four points, are of order $(\omega\beta)^{-3}$.

Finally the C′ diagrams are triangles $[(\omega\beta)^{-2}]$ with an extra end and are thus of order $(\omega\beta)^{-3}$.

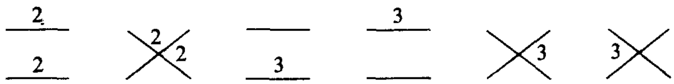
To order $(\omega\beta)^{-2}$ we may write

$$\int (1 + \Delta_1)^{-3/2}(1 + \Delta_2)^{-3/2}\Delta_{12}^2\, d\tau$$
$$= 2 \int \Delta_{12}^2[(1 + \Delta_1)^{-3/2} - 1]\, d\tau + \int \Delta_{12}^2\, d\tau \quad \text{(A.19)}$$

$$\int \Delta_{12}^2(1 + \Delta_1)^{-3/2}\, d\tau = \frac{4}{\omega\beta} - \frac{16 \ln 2}{\omega^2\beta^2} \tag{A.20}$$

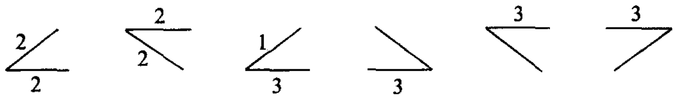
$$\int \Delta_{12}^2[(1 + \Delta_1)^{-3/2} - 1]\, d\tau = 8(\omega\beta)^{-2}(2 - 2 \ln 2) \tag{A.21}$$

Let us now examine the terms contained in $\int \Delta_{12}^4\, d\tau$. We have established that the order of $\omega\beta$ does not depend on the number of lines between two points. In addition, connected diagrams involving all four points are of order $(\omega\beta)^{-3}$. A diagram that contributes to order $(\omega\beta)^{-1}$ is —4—, with three others of the same type. The $(\omega\beta)^{-2}$ diagrams are the disconnected diagrams



(A.22)

We also have contributions of order $(\omega\beta)^{-2}$ from



(A.23)

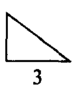
We find

$$\int \Delta_{12}^4\, d\tau = \frac{2}{\omega\beta} + \frac{44}{3}\frac{1}{(\omega\beta)^3} \tag{A.24}$$

The same type of diagrams contribute to $\int \Delta_{12}^n\, d\tau$, for $n > 2$.

We note that terms of type $\int \Delta_1\Delta_{12}^4\, d\tau$ involve an extra vertical line. So there are contributions from the four diagrams of the form

and also from connected diagrams of the form



We note that $\int \Delta_{12}^2(1 + \Delta_1)^{-3/2}\, d\tau$ had an $(\omega\beta)^{-2}$ term that was ln 2 times that from the first term in the expansion, viz., $\int \Delta_{12}^2\, d\tau$. We expect a similar modification when we compute $\int \Delta_{12}^4(1 + \Delta_1)^{-5/2}\, d\tau$. We do not compute it since the $\int \Delta_{12}^4\, d\tau$ contribution to the prefactor is only a few percent.

NOTE ADDED IN PROOF

Samathiyakanit[27] has used the two-time action as presented in this paper and obtained some of the results found in Section 3. However, he then discusses topics different from the present paper.

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